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Research Article

Equation of Existence of Motion of Three Body and Four Body Problem

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ABSTRACT

The work establishes the study of Sitnikov problem when extended to four body problem when all the primaries are point masses. Here all the primaries are moving in the circular orbit in the XY-plane and the infinitesimal particles that is third or fourth particle is moving along the z-axis. This work consists of the basic concept of celestial mechanics, Sitnikov problem and its extension to four body problem. In most of the part of this work, we study the Sitnikov problem. Also a little work is done about the four body problem and shown that how it is extended to four body problems.

Keywords: Sitnikov Problem, Celestial Mechanics, Three Body, Four Body

Introduction

Celestial Mechanics is a branch of Astronomy that is concerned with the motions of celestial objects-In particular, the objects that make up the solar system. The main aim of celestial mechanics is to reconcile these motions with the predications of Newtonian Mechanics. Examples include 3-body problem and 4-body diagram.

The framework comprises of two essential bodies with a similar mass (m_1=m_1=m/2),which move in round or curved Kepler's circles around their focal point of mass. Third body, which is generously littler than the essential bodies and whose mass can set to zero (m3=0), moves affected by the essential bodies in a plane that is opposite to the orbital plane of the essential bodies In such a framework, that the third body just moves in a single measurement. It moves just along the z-pivot.

The Sitnikov issue is a limited adaptation of the three-body issue named after Russian Mathematician Kirill Alex and raovitch Sitnikov that endeavors to portray the development of the three divine bodies because of their shared gravitational fascination. An exceptional instance of the Sitnikov issue was first found by the American researcher William Duncan Macillan in 1911, yet the issue as it as of now stands wasn't found until 1961 by Sitnikov.

Research Methodolgy

Equation of Motion of Three Body Derived from Newton's Law of Gravity

$$\overrightarrow{F_{i,j}} = -G \sum_{i \neq j,\ j=1}^{3} m_i\, m_j \frac{\overrightarrow{(r_i} - \vec{r}_j)}{\Delta_{i,j}^3}$$

Where, G is the gravitational constant. If we denote the accelerations by with k =1, 2, 3 then the system of equations of motion is given by:

$$\ddot{\vec{r}_1} = -G\left(\frac{m_2\overrightarrow{(r_1} - \vec{r}_2)}{\Delta_{1,2}^3} + \frac{m_3\overrightarrow{(r_1} - \vec{r}_3)}{\Delta_{1,3}^3}\right)$$

$$\ddot{\vec{r}_2} = -G\left(\frac{\overrightarrow{m_1(\vec{r}_2-\vec{r}_1)}}{\Delta^3_{1,2}} + \frac{\overrightarrow{m_3(\vec{r}_2-\vec{r}_3)}}{\Delta^3_{2,3}}\right)$$

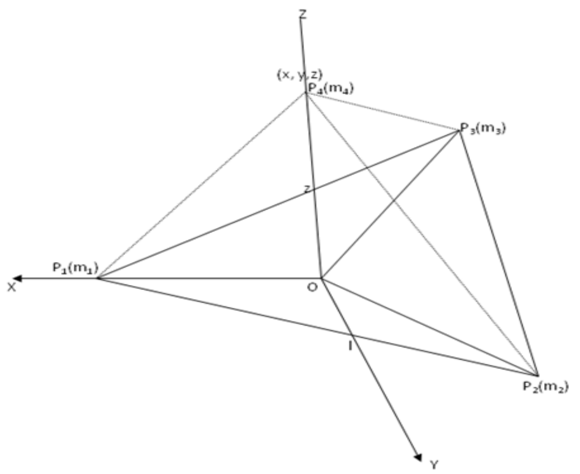
$$\ddot{\vec{r}_3} = -G\left(\frac{\overrightarrow{m_1(\vec{r}_3-\vec{r}_1)}}{\Delta^3_{1,3}} + \frac{\overrightarrow{m_2(\vec{r}_3-\vec{r}_2)}}{\Delta^3_{2,3}}\right)$$

Then, the anti-symmetry between the unperturbed motions of P_1, P_2 is such that, $\vec{r_1} = \vec{r_2} = (x_1, x_2, 0)$ where we have already set $z_1 = 0$ that is consistent with the assumptions that we make on the third body.

Equation of Motion of Four Body

Assume P_1, P_2 and P_3 be the 3 primaries of equal masses $m_1=m_2=m_3=m$(say). The fourth body P has a mass m_4 which is much less than the masses of the primaries. All the primaries are at the vertices of an equilateral triangle $P_1P_2P_3$ (McCuskey 1963) and are moving in a common circular or bit around their common center of mass O which is taken as origin and $P_1P_2=P_2P_3=P_3P_1=l$ as the distance between only two consecutive primaries. The infinitesimal mass m_4 is moving along a straight line perpendicular to the plane of motion of the primaries and passing through the center of mass of the primaries. In such a system the motion of the infinitesimal mass is one dimensional. The line OP_1 be considered as the x-axis, then the coordinates of the primaries are given by.

$$P_1\left(\frac{l}{\sqrt{3}},0,0\right), P_2\left(-\frac{l}{2\sqrt{3}},\frac{l}{2},0\right) \& P_3\left(-\frac{1}{2\sqrt{3}},-\frac{l}{2},0\right)$$



Configuration

Let at time t, (x, y, z) be the coordinates of the infinitesimal mass with respect to the frame (O,XYZ), the total forces ac ting on the infinitesimal mass m_4 due to the primaries, the position vector of the infinitesimal mass and be the angular velocity of the primaries about their common centre of mass O, then the equation of motion of the infinitesimal mass is:

$$\left\{\frac{\partial^2\vec{r}}{\partial t^2} + 2\vec{\omega}\times\frac{\partial\vec{r}}{\partial t} + \vec{\omega}\times(\vec{\omega}\times\vec{r})\right\} = \vec{F} \tag{1}$$

where, $\vec{\omega} = n\hat{k}$ is the unit vector perpendicular to the plane of motion of the primaries n is the mean motion of the primaries about O.

$$\left.\begin{aligned} \vec{F} &= -Gm_4\left(\frac{m_1}{r^3_{41}}\vec{r}_{41} + \frac{m_2}{r^3_{42}}\vec{r}_{42} + \frac{m_3}{r^3_{43}}\vec{r}_{43}\right) \\ \vec{r} &= x\,\hat{i} + y\,\hat{j} + z\,\hat{k} \\ \vec{r}_{41} &= \left(x - \frac{l}{\sqrt{3}}\right)\hat{i} + y\,\hat{j} + z\,\hat{k} \\ \vec{r}_{42} &= \left(x + \frac{l}{\sqrt{3}}\right)\hat{i} + \left(y - \frac{l}{2}\right)\hat{j} + z\,\hat{k} \\ \vec{r}_{43} &= \left(x + \frac{l}{2\sqrt{3}}\right)\hat{i} + \left(y + \frac{l}{2}\right)\hat{j} + z\,\hat{k} \end{aligned}\right\} \quad (2),\ (3)$$

Combining Eqs. (1), (2) and (3) and comparing the coefficeints of $\hat{i}$,$\hat{j}$ and $\hat{k}$, we get

$$\left.\begin{aligned} \ddot{x} - 2n\dot{y} &= n^2x - G\left\{\frac{m_1}{r^3_{41}}\left(x-\frac{l}{\sqrt{3}}\right) + \frac{m_2}{r^3_{42}}\left(x+\frac{l}{2\sqrt{3}}\right) + \frac{m_3}{r^3_{43}}\left(x+\frac{l}{2\sqrt{3}}\right)\right\} \\ \ddot{y} - 2n\dot{x} &= n^2y - G\left\{\frac{m_1}{r^3_{41}} + \frac{m_2}{r^3_{42}}\left(y-\frac{l}{2}\right) + \frac{m_3}{r^3_{43}}\left(y+\frac{l}{2}\right)\right\} \\ \ddot{z} &= -G\left\{\frac{m_1}{r^3_{41}} + \frac{m_2}{r^3_{42}} + \frac{m_3}{r^3_{43}}\right\} \end{aligned}\right\} \tag{4}$$

Following the method of Pandey and Ahmad (2013) the mean motion 'n' of the primaries can be determine as The acceleration of any point mass moving in a circular orbit is given by $\vec{a_i} = -r_i n^2 \hat{r}_i, r_i = |\vec{r}_i|, \hat{r}_i$ is the unit vector (i=1,2,3).

The equations of motion of three masses can be written as

$$m_1\ddot{\vec{r_1}} = -Gm_1\left\{\frac{m_2}{r^3_{12}}\overrightarrow{r_{12}} + \frac{m_3}{r^3_{13}}\overrightarrow{r_{13}}\right\}$$

$$m_2\ddot{\vec{r_2}} = -Gm_2\left\{\frac{m_3}{r^3_{23}}\overrightarrow{r_{12}} - \frac{m_1}{r^3_{12}}\overrightarrow{r_{12}}\right\}$$

$$m_3\ddot{\vec{r_3}} = -Gm_3\left\{\frac{m_2}{r^3_{23}}\overrightarrow{r_{23}} - \frac{m_3}{r^3_{13}}\overrightarrow{r_{13}}\right\}$$

The above set of equations can be written as:

$$-r_1n^2\hat{r}_1 = G\left\{-\frac{m_2(r_2\hat{r}_2 - r_1\hat{r}_1)}{r^3_{12}} + \frac{m_3(r_3\hat{r}_3 - r_1\hat{r}_1)}{r^3_{13}}\right\} \tag{5}$$

$$-r_1n^2\hat{r}_2 = G\left\{-\frac{m_3(r_3\hat{r}_3 - r_2\hat{r}_2)}{r^3_{23}} - \frac{m_1(r_2\hat{r}_2 - r_1\hat{r}_1)}{r^3_{12}}\right\} \tag{6}$$

$$-r_3n^2\hat{r}_3 = G\left\{-\frac{m_2(r_3\hat{r}_3 - r_2\hat{r}_2)}{r^3_{23}} - \frac{m_1(r_3\hat{r}_3 - r_1\hat{r}_1)}{r^3_{13}}\right\} \tag{7}$$

Since O is the center of mass of the system, we have

$$m_1\vec{r_1} + m_2\vec{r_2} + m_3\vec{r_3} = 0 \tag{8}$$

Multiplying the Eqs. (5) by, (6) by, adding them and using the Eq. (8), one can find

$$-r_3n^2\hat{r}_3 = G\left\{-\frac{m_2(r_3\hat{r}_3 - r_2\hat{r}_2)}{r^3_{23}} - \frac{m_1(r_3\hat{r}_3 - r_1\hat{r}_1)}{r^3_{13}}\right\}$$

Since the above equation and Eq. (7) are same hence Eq.(8) can be used in place of the Eq. (7). By re-arranging the equations (5), (6) and (8), one can find.

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$$\left(-n^2+\frac{Gm_2}{r_{12}^3}+\frac{Gm_3}{r_{13}^3}\right)r_1\hat{r}_1-\frac{Gm_2}{r_{12}^3}r_2\hat{r}_2-\frac{Gm_3}{r_{13}^3}r_3\hat{r}_3=0 \quad (9)$$

$$\left(-\frac{Gm_1}{r_{21}^3}\right)r_1\hat{r}_1+\left(-n^2+\frac{Gm_1}{r_{21}^3}+\frac{Gm_3}{r_{23}^3}\right)r_2\hat{r}_2-\frac{Gm_3}{r_{23}^3}r_3\hat{r}_3=0 \quad (10)$$

$$m_1r_1\hat{r_1}+m_2r_2\hat{r_2}+m_3r_3\hat{r_3}=0 \quad (11)$$

In polar system the unit vector may be expressed as

$\hat{r_1}\widehat{r_{h1}}=\cos\theta_h\hat{\imath}+\sin\theta_h\hat{\jmath}h$ = 1,2,3

Where, are the orientations of the position vectors of the primaries with respect to the axes. Using the above transformation the Equation (9), (10) and (11) becomes

$$\left(-n^2+\frac{Gm_2}{r_{12}^3}+\frac{Gm_3}{r_{13}^3}\right)r_1(\cos\theta_1\hat{\imath}+\sin\theta_1\hat{\jmath})-\left(\frac{Gm_1}{r_{12}^3}\right)r_1$$
$$(\cos\theta_2\hat{\imath}+\sin\theta_2\hat{\jmath})-\frac{Gm_3}{r_{13}^3}(\cos\theta_3\hat{\imath}+\sin\theta_3\hat{\jmath})=0 \quad (12)$$

$$-\left(\frac{Gm_1}{r_{21}^3}\right)r_1(\cos\theta_1\hat{\imath}+\sin\theta_1\hat{\jmath})+\left(-n^2+\frac{Gm_1}{r_{21}^3}+\frac{Gm_3}{r_{23}^3}\right)r_2$$
$$(\cos\theta_2\hat{\imath}+\sin\theta_2\hat{\jmath})-\left(\frac{Gm_3}{r_{23}^3}\right)r_3(\cos\theta_3\hat{\imath}+\sin\theta_3\hat{\jmath})=0 \quad (13)$$

$$m_1r_1(\cos\theta_1\hat{\imath}+\sin\theta_1\hat{\jmath})+m_2r_2(\cos\theta_2\hat{\imath}+\sin\theta_2\hat{\jmath})+m_3r_3$$
$$(\cos\theta_3\hat{\imath}+\sin\theta_3\hat{\jmath})=0 \quad (14)$$

The Eqs. (12), (13) and (14) can be written in Cartesian form as

$$\left.\begin{aligned}&\left(-n^2+\frac{Gm_2}{r_{12}^3}+\frac{Gm_3}{r_{13}^3}\right)x_1-\frac{Gm_2}{r_{12}^3}x_2-\frac{Gm_3}{r_{13}^3}x_3=0\\&\left(-\frac{Gm_1}{r_{21}^3}\right)x_1+\left(-n^2+\frac{Gm_1}{r_{21}^3}+\frac{Gm_3}{r_{23}^3}\right)x_2-\frac{Gm_3}{r_{23}^3}x_3=0\end{aligned}\right\} \quad (15)$$

$$\left.\begin{aligned}&\left(-n^2+\frac{Gm_2}{r_{12}^3}+\frac{Gm_3}{r_{13}^3}\right)y_1-\frac{Gm_2}{r_{12}^3}y_2-\frac{Gm_3}{r_{13}^3}y_3=0\\&\frac{Gm_1}{r_{21}^3}y_1+\left(-n^2+\frac{Gm_1}{r_{21}^3}+\frac{G_{m_3}}{r_{23}^3}\right)y_2-\frac{Gm_3}{r_{23}^3}y_3=0\\&m_1y_1+m_2y_2+m_3y_3=0\end{aligned}\right\} \quad (16)$$

Let m_1, m_2 and m_3 be the vertices of an equilateral triangle, then $r_{12}+r_{23}+r_{31}=l$ (say).

The Eqs. (15) and (16) have non-trivial solution if

$$\begin{vmatrix}\left(-n^2+\frac{2Gm}{l^3}\right) & -\frac{Gm}{l^3} & -\frac{Gm}{l^3} & 0 & 0 & 0\\ -\frac{Gm}{l^3} & \left(-n^2+\frac{2Gm}{l^3}\right) & -\frac{Gm}{l^3} & 0 & 0 & 0\\ m & m & m & 0 & 0 & 0\\ 0 & 0 & 0 & \left(-n^2+\frac{2Gm}{l^3}\right) & -\frac{Gm}{l^3} & -\frac{Gm}{l^3}\\ 0 & 0 & 0 & -\frac{Gm}{l^3} & \left(-n^2+\frac{2Gm}{l^3}\right) & -\frac{Gm}{l^3}\\ 0 & 0 & 0 & m & m & m\end{vmatrix}=0$$

On Simplification, one can find

$$m^2n^8-\frac{12Gm^3n^6}{l^3}+\frac{54G^2m^4n^4}{l^6}+\frac{108G^3m^5n^2}{l^6}+\frac{54G^4m^6}{l^{12}}=0$$

$$m^2(3Gm-n^2l^3)$$

$$\frac{m(3Gm-n^2l^3)^4}{l^{12}}=0$$

$$n^2=\frac{3Gm}{l^3} \quad (17)$$

Here, let us fix the unit of time by G=1, unit of distance by l=1 and unit of masses by m=1/3, then Eq. (17) gives n=1 and consequently the set of Eqs. (4) became

$$\left.\begin{aligned}&\ddot{x}-2\dot{y}=\Omega_x\\&\ddot{y}-2\dot{x}=\Omega_y\\&\ddot{z}-\Omega_z\end{aligned}\right\} \quad (18)$$

Where the force function Ω is given by

$$\Omega=\frac{1}{2}(x^2+y^2)+\frac{1}{3}\sum_{i=1}^{3}\frac{1}{r_{4i}}$$

In the problem, the motion of is only along the z-axis hence the equation of motion of the four-body problem is obtained from the Eqs. (18) by putting x(0)=y(0)=0, we have

$$\frac{d^2z}{dt^2}=-\frac{z}{(z^2+\frac{1}{3})^{3/2}}$$

Result and Discussion

Let us assume that 1) the mass of the third body can be neglected with respect to the masses of the primary bodies, and 2) that the motion of the third body starts along the z-axis with special initial conditions, such that the velocity components of P_3 that are normal to the z-axis vanish.

we formally set $m_3=0$, $x_3(0)=y_3(0)=0$, $\dot{x}_3(0)=\dot{y}_3(0)=0$ and $\ddot{x}_3(0)=\ddot{y}_3(0)=0$ thus find.

The assumptions we make allow us to investigate the reduced set of differential equation:

$$\ddot{x}_3(0)=-\frac{Gmx_1}{(4x_1{}^2+4y_1{}^2)^{3/2}}$$

$$\ddot{y}_3(0)=-\frac{Gmy_1}{(4x_1{}^2+4y_1{}^2)^{3/2}}$$

$$\ddot{z}_3(0)=-\frac{2Gmz_3}{(x_1{}^2+y_1{}^2+z_3{}^2)^{3/2}}$$

We notice that the first two of the above equations are uncoupled form the third. Moreover, the coupling terms in the third equation are of the special form $r=x_1{}^2+y_1{}^2$. that is the distance of one of the primaries from the common barycenter of the system.

Expressing the radial component of unperturbed motion in terms of

$$\ddot{z}=-\frac{2Gmz}{(r^2+z^2)^{3/2}}$$

If we then substitute for r = r(t), we are left with:

$$\ddot{z}=-\frac{2Gmz}{(r(t)^2+z^2)^{3/2}}=0$$

that is a dimensional $1\frac{1}{2}$, non-autonomous equation of motion that describes the restricted motion of the third body along the z-axis under our assumptions that we made.

Conclusion

The motion of the third body along the z-axis is that is

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$\ddot{z} = -\frac{2Gmz}{(r(t)^2+z^2)^{3/2}}$a $1\frac{1}{2}$, dimensional, non-autonomous equation of motion.

The motion of the forth body along z-axis is $\frac{d^2z}{dt^2} = -\frac{z}{(z^2+\frac{1}{3})^{3/2}}$ that is one dimensional equation of motion.

The infinitesimal mass i.e. m_3 and m_4 lies on z-axis and is perpendicular to the orbital plane of the primary bodies.

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